

Rigid spheres in Riemannian spaces

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March 8, 2013

Abstract

Choice of an appropriate (3+1)-foliation of spacetime or a (2+1)-foliation of the Cauchy space, leads often to a substantial simplification of various mathematical problems in General Relativity Theory. We propose a new method to construct such foliations. For this purpose we define a special family of topological two-spheres, which we call “rigid spheres”. We prove that there is a four-parameter family of rigid spheres in a generic Riemannian three-manifold (in case of the flat Euclidean three-space these four parameters are: 3 coordinates of the center and the radius of the sphere). The rigid spheres can be used as building blocks for various (“spherical”, “bispherical” etc.) foliations of the Cauchy space. This way a supertranslation ambiguity may be avoided. Generalization to the full 4D case is discussed. Our results generalize both the Huang foliations (cf. [4]) and the foliations used by us (cf. [8]) in the analysis of the two-body problem.

1 Introduction

The choice of an appropriate coordinate system may sometimes be decisive in the analysis of dynamical properties of a physical system. Symmetries of the system provide here an important hint. Unfortunately, a generic configuration of gravitational field in general relativity theory admits no symmetries. But still the nature of the problem suggests very often at least a topological character of the coordinate system. As an example consider the quasi-local mass which, according to Penrose (cf. [10]), can be assigned to a topological two-sphere $S \subset \Sigma$, where Σ is a Riemannian 3D manifold, playing the role of a Cauchy surface for Einstein equations (cf. [10]). Here, topologically spherical coordinates, such that $S = \{r = \text{const.}\}$ for a certain radial coordinate r , enable us to simplify considerably the formulation of the theory. Recently we have proposed a new approach to the two-body problem, where topologically bi-spherical coordinates are extensively used to describe the gravitational field surrounding two black holes (cf. [8]). In both cases, an appropriate family of two-surfaces, homeomorphic to S^2 , has to be chosen as an important geometric ingredient of the theory.

A possible construction of such a family consists e.g. in imposing an elliptic (cf. [2]) or a parabolic (cf. [5], [6]) partial differential equation for the radial coordinate r . An obvious drawback of such an approach consists in the fact that we do not control intrinsic properties of the surfaces $\{r = \text{const.}\}$ constructed this way.

In the present paper we propose an alternative approach, based on intrinsic geometric properties of special two-spheres. Such objects can be later used as building blocks for various foliations. We mimic here the properties of the four-parameter family of metric spheres in the Euclidean three-space E^3 , which are uniquely characterized by a simple requirement: $k = \text{const.}$, where k denotes the mean extrinsic curvature. In a generic Riemannian three-manifold, the above equation is, however, too restrictive and admits a single solution, related to the “center of mass” of the geometry (cf. [4]), whereas in the flat case it admits a four-parameter family of solutions. This proves that the condition is not stable with respect to small perturbations of the geometry. Moreover, even in the (flat) Euclidean space this condition may be replaced by a weaker one, which still selects the same family of spheres. We show in the present paper that an analogous, weaker than $k = \text{const.}$, condition admits a four-parameter family of solutions in a generic non-flat case, provided the geometry of the space does not differ too much from the Euclidean geometry. Topological two-spheres satisfying this condition will be called “rigid spheres”. We expect that foliating the Cauchy three-surface Σ with various families of rigid spheres, we will be able to construct useful gauge conditions in general relativity theory, similarly as in [8].

Our construction is based on the following idea. Take any surface S satisfying the rigid sphere condition and consider infinitesimal deformations of S . They may be parameterized by sections of the normal bundle $T^\perp S$. If we want our condition

to admit a four-parameter family of solutions, like in the flat case, its linearization must admit a four-parameter family of deformations. This means that we are not allowed to constrain the complete information about the mean curvature k : four real parameters describing k must be left free. In the flat case these four parameters which have to be left free are: the mean value (or the *monopole part*) of k , which is responsible for the size of S , and its *dipole part* which vanishes due to Gauss-Codazzi equations. The *dipole part* of the deformation is related to the group of translations. In fact, possible motions of metric spheres are described by the group of Euclidean motions, quotiented by the subgroup of rotations which are internal symmetries of every particular sphere S .

To implement the above idea, an intrinsic, geometric notion of a multipole expansion on an arbitrary Riemannian two-surface which is topologically S^2 is proposed in Section 2 and a notion of a rigid sphere is then defined. In Section 3 we prove that a four-parameter family of rigid spheres exists in a generic Riemannian three-space.

The present paper is a part of a bigger project, where an eight-parameter family of similar “rigid spheres” is proven to exist in a generic four-dimensional Lorenzian spacetime. In the present paper we limit ourselves to the case of a Riemannian three-manifold, but our construction can be generalized to the entire pseudo-Riemannian spacetime M , instead of the Riemannian Cauchy three-space $\Sigma \subset M$. The idea of this extension is to mimic the case of the flat Minkowski space, where all possible round spheres, embedded in all possible flat subspaces Σ of M , form an eight-parameter family. All of them can be obtained from a single one by the action of the product of the one-parameter group of dilations (changing the size of S) and the ten-parameter Poincaré group, quotiented by the three-parameter rotation group. In this case, it is necessary to take into account not only the external curvature of S , but also its torsion. The rigid spheres obtained this way form an eight-parameter family and may be used to construct useful coordinate systems not only on a given Cauchy surface Σ , but also in the entire spacetime. The main advantage of such a construction consists in its rigidity at infinity: it eliminates supertranslations and reduces the symmetry group of the Scri, otherwise infinite dimensional, to the finite-dimensional one. The corresponding result will be presented in a subsequent paper.

2 Equilibrated spherical coordinates

2.1 Conformally spherical coordinates

Let S be a differential two-manifold, diffeomorphic to the two-sphere $S^2 \subset \mathbb{R}^3$ and equipped with a (sufficiently smooth) metric g . Coordinates $(\vartheta, \varphi) = (x^A)$, $A = 1, 2$, defined on a dense subset of $S \setminus \ell$, where ℓ is topologically a line interval, will be called *conformally spherical coordinates* if they have the same range of

values as the standard spherical coordinates on $S^2 \subset \mathbb{R}^3$ and, moreover, if the corresponding metric tensor g_{AB} is conformally equivalent to the standard round metric on S^2 , i.e. the following formula holds:

$$g_{AB} = \psi \cdot \sigma_{AB} , \quad (1)$$

where ψ is a (sufficiently smooth) function on S and

$$\sigma_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \vartheta \end{pmatrix} . \quad (2)$$

Remark 1. Conformally spherical coordinates always exist (cf. [9]). It is easy to see that there is always a six-parameter freedom in the choice of such coordinates. More precisely: if (ϑ, φ) are conformally spherical coordinates then $(\tilde{\vartheta}, \tilde{\varphi})$ are also conformally spherical if and only if they may be obtained from (ϑ, φ) *via* a conformal transformation of $S^2 \subset \mathbb{R}^3$.

Example 1. A “proper” conformal transformation, i. e. which is not a rotation: Let $\mathbf{n} \in S$ and $\tau > 0$ be a positive number. Using appropriate rotation, choose conformally spherical coordinates (ϑ, φ) in such a way that \mathbf{n} is a north pole, i.e. the coordinate ϑ vanishes at \mathbf{n} . Define

$$F_{\mathbf{n},\tau}(\vartheta, \varphi) = (\tilde{\vartheta}, \tilde{\varphi}) , \quad (3)$$

where

$$\tilde{\vartheta} := 2 \arctan \left(\tau \cdot \tan \frac{\vartheta}{2} \right) , \quad \tilde{\varphi} := \varphi , \quad (4)$$

or, equivalently,

$$\tan \frac{\tilde{\vartheta}}{2} = \tau \cdot \tan \frac{\vartheta}{2} . \quad (5)$$

For the fixed point \mathbf{n} these transformations form a one-parameter group:

$$F_{\mathbf{n},\tau} \circ F_{\mathbf{n},\sigma} = F_{\mathbf{n},\tau\sigma} , \quad (6)$$

generated by the vector field:

$$\left. \frac{d}{dt} \right|_{t=1} F_{\mathbf{n},t}(\vartheta, \varphi) = \left. \frac{d}{dt} \right|_{t=1} \left[2 \arctan \left(t \cdot \tan \frac{\vartheta}{2} \right) \right] \frac{\partial}{\partial \vartheta} = \sin \vartheta \frac{\partial}{\partial \vartheta} . \quad (7)$$

In particular, $F_{\mathbf{n},1} = \mathbb{I}$ (the identity map) for every \mathbf{n} . Moreover, equation (5) implies the following identity:

$$F_{-\mathbf{n},\tau} = F_{\mathbf{n},\frac{1}{\tau}} . \quad (8)$$

Using (4) and (5) we may easily derive the following formula:

$$d\vartheta = \frac{d\tilde{\vartheta}}{d\vartheta} d\tilde{\vartheta} = \tau \frac{1 + \tan^2 \frac{\tilde{\vartheta}}{2}}{\tau^2 + \tan^2 \frac{\tilde{\vartheta}}{2}} d\tilde{\vartheta} . \quad (9)$$

Similarly, we may prove:

$$\sin \vartheta = \frac{\sin \vartheta}{\sin \tilde{\vartheta}} \sin \tilde{\vartheta} = \frac{2 \tan \frac{\vartheta}{2}}{1 + \tan^2 \frac{\vartheta}{2}} \cdot \frac{1 + \tan^2 \frac{\tilde{\vartheta}}{2}}{2 \tan \frac{\tilde{\vartheta}}{2}} \sin \tilde{\vartheta} \quad (10)$$

$$= \frac{2 \frac{1}{\tau} \tan \frac{\tilde{\vartheta}}{2}}{1 + \frac{1}{\tau^2} \tan^2 \frac{\tilde{\vartheta}}{2}} \cdot \frac{1 + \tan^2 \frac{\tilde{\vartheta}}{2}}{2 \tan \frac{\tilde{\vartheta}}{2}} \sin \tilde{\vartheta} = \tau \frac{1 + \tan^2 \frac{\tilde{\vartheta}}{2}}{\tau^2 + \tan^2 \frac{\tilde{\vartheta}}{2}} \sin \tilde{\vartheta} . \quad (11)$$

As a conclusion we obtain:

$$(\mathrm{d}\vartheta)^2 + \sin^2 \vartheta (\mathrm{d}\varphi)^2 = h^2 \left[(\mathrm{d}\tilde{\vartheta})^2 + \sin^2 \tilde{\vartheta} (\mathrm{d}\varphi)^2 \right] , \quad (12)$$

where

$$h = \tau \frac{1 + \tan^2 \frac{\tilde{\vartheta}}{2}}{\tau^2 + \tan^2 \frac{\tilde{\vartheta}}{2}} , \quad (13)$$

which proves the conformal character of the transformation. Indeed, we have

$$g_{AB} \mathrm{d}x^A \mathrm{d}x^B = \psi \left[(\mathrm{d}\vartheta)^2 + \sin^2 \vartheta (\mathrm{d}\varphi)^2 \right] = \psi h^2 \left[(\mathrm{d}\tilde{\vartheta})^2 + \sin^2 \tilde{\vartheta} (\mathrm{d}\varphi)^2 \right] . \quad (14)$$

Hence, $(\tilde{\vartheta}, \varphi)$ are conformally spherical coordinates if (ϑ, φ) were.

2.2 A barycenter of a conformally spherical system

Given a system of conformally spherical coordinates on S , consider the corresponding three functions:

$$x := \sin \vartheta \cos \varphi , \quad (15)$$

$$y := \sin \vartheta \sin \varphi , \quad (16)$$

$$z := \cos \vartheta . \quad (17)$$

We have, therefore, a mapping $\mathbf{D}:]0, \pi[\times]0, 2\pi[\mapsto \mathbb{R}^3$, given by:

$$\mathbf{D}(\vartheta, \varphi) = \begin{pmatrix} D^1(\vartheta, \varphi) \\ D^2(\vartheta, \varphi) \\ D^3(\vartheta, \varphi) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \quad (18)$$

The following vector

$$\mathbf{X} = \begin{pmatrix} \langle x \rangle \\ \langle y \rangle \\ \langle z \rangle \end{pmatrix} \in \mathbb{R}^3 , \quad (19)$$

where by $\langle f \rangle$ we denote the average (mean value) of the function f on S , i.e. the number

$$\langle f \rangle := \frac{\int_S f \sqrt{\det g} \, \mathrm{d}^2 x}{\int_S \sqrt{\det g} \, \mathrm{d}^2 x} , \quad (20)$$

will be called a “barycenter” of the system (ϑ, φ) on S . Of course, we have $\|\mathbf{X}\| \leq 1$.

Example 2. Consider the proper conformal transformation (4) and calculate the new barycenter

$$\tilde{\mathbf{X}} = \begin{pmatrix} \langle \tilde{x} \rangle \\ \langle \tilde{y} \rangle \\ \langle \tilde{z} \rangle \end{pmatrix} \in \mathbb{R}^3, \quad (21)$$

where

$$\begin{aligned} \tilde{x} &:= \sin \tilde{\vartheta} \cos \varphi, \\ \tilde{y} &:= \sin \tilde{\vartheta} \sin \varphi, \\ \tilde{z} &:= \cos \tilde{\vartheta}. \end{aligned}$$

The trigonometric identity:

$$\cos \vartheta = \frac{1 - \tan^2 \frac{\vartheta}{2}}{1 + \tan^2 \frac{\vartheta}{2}}, \quad (22)$$

implies:

$$\tan^2 \frac{\vartheta}{2} = \frac{1 - \cos \vartheta}{1 + \cos \vartheta} = \frac{1 - z}{1 + z}. \quad (23)$$

Hence, the formula (5) implies:

$$\frac{1 - \tilde{z}}{1 + \tilde{z}} = \tau^2 \frac{1 - z}{1 + z}, \quad (24)$$

or, equivalently,

$$\tilde{z} = \frac{1 + z - \tau^2(1 - z)}{1 + z + \tau^2(1 - z)}. \quad (25)$$

Moreover, the formula (11) and its inverse:

$$\sin \tilde{\vartheta} = \tau \frac{1 + \tan^2 \frac{\vartheta}{2}}{1 + \tau^2 \tan^2 \frac{\vartheta}{2}} \sin \vartheta = \tau \frac{1 + \frac{1-z}{1+z}}{1 + \tau^2 \frac{1-z}{1+z}} \sin \vartheta = \frac{2\tau \sin \vartheta}{1 + z + \tau^2(1 - z)}, \quad (26)$$

imply:

$$\tilde{x} := \frac{2\tau}{1 + z + \tau^2(1 - z)} x, \quad (27)$$

$$\tilde{y} := \frac{2\tau}{1 + z + \tau^2(1 - z)} y. \quad (28)$$

To calculate mean value of the functions (27), (28) and (25) we do not need to pass to new coordinates $(\tilde{\vartheta}, \varphi)$, but we may use, as well, old coordinates (ϑ, φ) . But we see that for $\tau \rightarrow 0$ we have $\tilde{x} \rightarrow 0$, $\tilde{y} \rightarrow 0$, $\tilde{z} \rightarrow 1$. The Lebesgue theorem implies, therefore, that for $\tau \rightarrow 0$ we have

$$\tilde{\mathbf{X}} = \begin{pmatrix} \langle \tilde{x} \rangle \\ \langle \tilde{y} \rangle \\ \langle \tilde{z} \rangle \end{pmatrix} \longrightarrow \begin{pmatrix} \langle 0 \rangle \\ \langle 0 \rangle \\ \langle 1 \rangle \end{pmatrix} = \mathbf{n}. \quad (29)$$

2.3 Equilibrated spherical systems

Definition 1. Conformally spherical coordinate system (ϑ, φ) is called *equilibrated*, if its barycenter vanishes: $\mathbf{X} = 0 \in \mathbb{R}^3$.

Remark 2. If there are two equilibrated spherical systems on S then they are related by a rotation.

Theorem 1. *Each metric tensor on S admits a unique (up to rotations) equilibrated spherical system.*

Proof. Given a metric tensor g on S , choose first any system of conformally spherical coordinates (ϑ, φ) on S and consider the corresponding identification of its points with the points of $S^2 = \partial K(0, 1) \subset \mathbb{R}^3$. Consider now the mapping

$$\mathbb{R}^3 \supset K(0, 1) \ni \mathbf{N} \rightarrow \mathcal{F}(\mathbf{N}) \in K(0, 1) \subset \mathbb{R}^3 , \quad (30)$$

given by the following formula

$$\mathcal{F}(\mathbf{N}) := \tilde{\mathbf{X}}_{\mathbf{n}, \tau} , \quad (31)$$

where the latter is the barycenter of the coordinates $(\tilde{\vartheta}, \tilde{\varphi})$ obtained from (ϑ, φ) by the proper conformal transformation (3) with

$$\mathbf{n} := \frac{\mathbf{N}}{\|\mathbf{N}\|} \quad (32)$$

and

$$\tau := 1 - \|\mathbf{N}\| . \quad (33)$$

Obviously, \mathcal{F} is continuous. Moreover, due to (29), it reduces to the identical mapping when restricted to the boundary $S^2 = \partial K(0, 1) \subset K(0, 1)$ (i.e. we have $\mathcal{F}(\mathbf{N}) = \mathbf{N}$ for $\|\mathbf{N}\| = 1$). This implies that there must be a point \mathbf{N}_0 which solves equation $\mathcal{F}(\mathbf{N}_0) = 0$. This completes the existence proof. To prove the uniqueness, let us suppose that there is another solution: $\mathcal{F}(\mathbf{N}_1) = 0$. Consider now the conformal transformation $F_{\mathbf{n}_1, \tau_1} \circ F_{\mathbf{n}_0, \tau_0}^{-1}$. Since the proper conformal transformations do not form any subgroup of the group of all conformal transformations, we cannot assume that it is again a proper transformation. But it may be decomposed into a product of rotations and a proper conformal transformation:

$$F_{\mathbf{n}_1, \tau_1} \circ F_{\mathbf{n}_0, \tau_0}^{-1} = \mathcal{O}_1 \circ F_{\mathbf{m}, \tau} \circ \mathcal{O}_0 , \quad (34)$$

where \mathcal{O}_1 and \mathcal{O}_0 are rotations. Denote by (ϑ_0, φ_0) the spherical coordinates obtained from (ϑ, φ) by the transformation $F_{\mathbf{n}_0, \tau_0}$ and then rotation \mathcal{O}_0^{-1} . Similarly, denote by (ϑ_1, φ_1) the ones obtained from (ϑ, φ) by $F_{\mathbf{n}_1, \tau_1}$ and then by rotation \mathcal{O}_1^{-1} . Because a rotation does affect equilibration of coordinates, both systems (ϑ_0, φ_0) and (ϑ_1, φ_1) are equilibrated. But the latter may be obtained from the

former by a proper conformal transformation $F_{\mathbf{m},\tau}$. We shall prove that this is impossible unless $\tau = 1$ or, equivalently, transformation $F_{\mathbf{m},\tau}$ is trivial (identical). For this purpose consider, for each value of τ , the linear function z_τ . Without any loss of generality we may assume that

$$\mathbf{m} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (35)$$

(if this is not the case, it is sufficient to perform an appropriate rotation of coordinates). Formula (25) implies the following relation:

$$z_\tau = \frac{1 + z_0 - \tau^2(1 - z_0)}{1 + z_0 + \tau^2(1 - z_0)} . \quad (36)$$

Hence

$$\frac{d}{dt} z_\tau = \frac{-4\tau(1 - z_0^2)}{[1 + z_0 + \tau^2(1 - z_0)]^2} \leq 0 , \quad (37)$$

and it vanishes only at a single point $z_0 = 1$. Consequently, its mean value:

$$< \frac{d}{dt} z_\tau > \quad (38)$$

is strictly negative. This implies that starting from $\tau = 1$ (which corresponds to the identity mapping $F_{\mathbf{m},1}$) and moving towards the actual value $\tau < 1$, the “ z ”-component of the vector $\tilde{\mathbf{X}}_{\mathbf{m},\tau}$ is strictly increasing. It vanishes at the beginning because (ϑ_0, φ_0) is equilibrated. Hence, it must be strictly positive at the end. This means that the final system (ϑ_1, φ_1) cannot be equilibrated unless $\tau = 1$ and, therefore, both systems coincide. \square

Different equilibrated spherical systems of coordinates form, therefore, a three-dimensional family. They can be parameterized by the position of a fixed point $\mathbf{n} \in S$ (north pole) and the geographic longitude of a fixed point $\mathbf{m} \in S$ (Greenwich). More precisely: given two points $\mathbf{n}, \mathbf{m} \in S$, $\mathbf{n} \neq \mathbf{m}$, there is a unique equilibrated spherical system (ϑ, φ) of coordinates on S , such that ϑ vanishes at \mathbf{n} and φ vanishes at \mathbf{m} .

Combining these observations with classical results (cf. [9]) we obtain the following

Theorem 2. *Let S be a differential two-manifold, diffeomorphic to the two-sphere $S^2 \subset \mathbb{R}^3$ and equipped with a metric g of class $C^{(k,\alpha)}$. For every pair $\mathbf{n}, \mathbf{m} \in S$, $\mathbf{n} \neq \mathbf{m}$, there is a unique equilibrated spherical system (ϑ, φ) of coordinates on S , such that ϑ vanishes at \mathbf{n} and φ vanishes at \mathbf{m} and the metric components g_{AB} are of the same class $C^{(k,\alpha)}$.*

Here, $C^{(k,\alpha)}$ is a Hölder space $C^{k,\alpha}(S^2)$, defined for $1 \leq k \in \mathbb{N}$ and $0 < \alpha < 1$. The space consists of those functions on S^2 which have continuous derivatives up to order k and such that the k -th partial derivatives are Hölder continuous with exponent α . This is a locally convex topological vector space.

The Hölder coefficient of a function f is defined as follows:

$$|f|_{C^{0,\alpha}} = \sup_{x,y \in S^2, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

The function f is said to be (uniformly) Hölder continuous with exponent α if $|f|_{C^{0,\alpha}}$ is finite. In this case the Hölder coefficient can be used as a seminorm.

The Hölder space $C^{k,\alpha}(S^2)$ is composed of functions whose derivatives up to order k are bounded and the derivatives of the order k are Hölder continuous. It is a Banach space equipped with the norm

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \max_{|\beta|=k} |D^\beta f|_{C^{0,\alpha}},$$

where β ranges over multi-indices and

$$\|f\|_{C^k} = \max_{|\beta| \leq k} \sup_{x \in S^2} |D^\beta f(x)|.$$

2.4 Rigid spheres in a Riemannian three-space

Given a manifold S equipped with a metric tensor g , there is a three-dimensional space of “linear functions” uniquely defined on S as linear combinations of functions (15–17), calculated in any equilibrated spherical system of coordinates (ϑ, φ) . We denote this space by \mathcal{M}^3 . By \mathcal{M}^4 we denote the space spanned by \mathcal{M}^3 and the constant functions on S . Linear functions (15–17) on S are eigenfunctions of the Laplace operator¹ Δ_σ , with the eigenvalue equal to -2 , i.e. $\Delta_\sigma X^i = -2X^i$, where we denote $x = X^1$, $y = X^2$, $z = X^3$. Let us denote by $d\sigma := \sin \vartheta d\vartheta d\varphi$ the measure associated with the metric σ_{AB} .

Definition 2. Let $f \in L^2(S, d\sigma)$. The projection of f onto the subspace of constant functions:

$$P_m(f) := \frac{1}{4\pi} \int_S f d\sigma \tag{39}$$

will be called the *monopole part* of f , whereas the projection onto $\mathcal{M}^3 = \text{span}\{X^1, X^2, X^3\}$:

$$P_d(f) := \sum_{i=1}^3 \left(X^i \frac{\int_S X^i f d\sigma}{\int_S (X^i)^2 d\sigma} \right) \tag{40}$$

¹By Δ_σ we denote the usual Laplace operator for the unit-sphere metric (2).

will be called the *dipole part* of f . In addition, we set

$$\mathcal{M}^4 := \text{span}\{1\} \oplus \mathcal{M}^3 = \text{span}\{1, X^1, X^2, X^3, \} \quad (41)$$

and $P_{md}(f) := P_m(f) + P_d(f) \in \mathcal{M}^4$ denotes the *mono-dipole part* of f .

The above structure enables us to define the multipole decomposition of the functions defined on a topological sphere S in terms of eigenspaces of the Laplace operator associated with the metric σ_{AB} . If h is a function on S then by $h^{\mathbf{m}} := P_m(h)$ we denote its monopole (constant) part, by $h^{\mathbf{d}} := P_d(h)$ — the dipole part (projection to the eigenspace with $l = 1$). By $h^{\mathbf{w}} := (I - P_{md})(h) = h - h^{\mathbf{m}} - h^{\mathbf{d}}$ we denote the “wave”, or mono-dipole-free, part of h , $h^{\mathbf{dw}} := (I - P_m)(h) = h - h^{\mathbf{m}} = h^{\mathbf{d}} + h^{\mathbf{w}}$, and finally $h^{\mathbf{md}} := P_{md}(h) = h^{\mathbf{m}} + h^{\mathbf{d}}$.

Remark 3. Mutually orthogonal projectors P_{md} and $P_w := (I - P_{md})$ are, of course, continuous, when considered as operators in the Hilbert space $L^2(S, d\sigma)$. For our purposes we have to consider them as operators in the Banach space $C^{(k, \alpha)}$. Here, no “orthogonality” is defined. Nevertheless, both operators are again continuous projectors. They define an isomorphism:

$$C^{(k, \alpha)} \cong C_{md}^{(k, \alpha)} \times C_w^{(k, \alpha)} ,$$

where $C_{md}^{(k, \alpha)} = P_{md}(C^{(k, \alpha)}) \equiv \mathcal{M}^4$ and $C_w^{(k, \alpha)} = P_w(C^{(k, \alpha)})$. Hence, a function $f \in C^{(k, \alpha)}$ is uniquely characterized by its mono-dipole part $f^{\mathbf{md}}$ and the remaining “wave” part $f^{\mathbf{w}}$, i.e. we have: $f = (f^{\mathbf{md}}, f^{\mathbf{w}})$.

Definition 3. Let Σ be a Riemannian three-manifold and let $S \subset \Sigma$ be a sub-manifold homeomorphic with $S^2 \subset \mathbb{R}^3$. We say that S is a *rigid sphere* if its mean extrinsic curvature k satisfies $k \in \mathcal{M}^4$, i.e. if the following equation holds:

$$k^{\mathbf{w}} = 0 . \quad (42)$$

2.5 The 4-D spacetime case – an outline

Definition of a rigid sphere in a Lorenzian four-manifold is more complicated: to control “rigidity” of a sphere, we must take into account more geometry. For this purpose we consider the extrinsic curvature vector of S : $k^a = k_{AB}^a g^{AB}$, where k_{AB}^a denotes the external curvature tensor of S (here, a, b are indices corresponding to the subspace orthogonal to S whereas A, B label coordinates on S). Moreover, we consider its torsion:

$$\ell_A = (\mathbf{m} | \nabla_A \mathbf{n}) , \quad (43)$$

where

$$\mathbf{n} := \frac{\mathbf{k}}{\|\mathbf{k}\|} , \quad (44)$$

$\|\mathbf{k}\| = \sqrt{k^a g_{ab} k^b}$, and \mathbf{m} is a vector orthogonal to both \mathbf{k} and S .

Definition 4. Let M be a Lorenzian four-manifold (a generic curved spacetime) and let $S \subset M$ be a *spacelike* submanifold homeomorphic with $S^2 \subset \mathbb{R}^3$. We say that S is a *rigid sphere* if $\mathbf{k} = (k^a)$ is spacelike and the following two conditions are satisfied:

$$\|\mathbf{k}\| \in \mathcal{M}^4, \quad (45)$$

$$\nabla_A \ell^A \in \mathcal{M}^3. \quad (46)$$

In this paper we limit ourselves to the purely Riemannian 3D-setting. The general, pseudo-riemannian case will be analyzed in a subsequent paper.

Example 3. *Rigid spheres in a four-dimensional Minkowski spacetime and in Euclidean three-space.*

Let M_0 be the flat Minkowski spacetime, i.e. the space \mathbb{R}^4 parameterized by the Lorentzian coordinates $(x^\alpha) = (x^0, \dots, x^3)$ and equipped with the metric $\eta = (\eta_{\alpha\beta}) = \text{diag}(-1, 1, 1, 1)$ (greek indices run always from 0 to 3). Consider in M_0 a *round sphere*, i.e. the two-dimensional submanifold defined by

$$S_{T,R} := \left\{ x \in \mathbb{R}^4 \mid x^0 = T, \sum_{i=1}^3 (x^i)^2 = R^2 \right\},$$

where the time $T \in \mathbb{R}$ and the sphere's radius $R > 0$ are fixed. It may be easily verified that the submanifold fulfills the following conditions:

$$\sqrt{k^a g_{ab} k^b} = \frac{2}{R} \in \mathcal{M}^4, \quad (47)$$

$$\nabla_A \ell^A = 0 \in \mathcal{M}^3, \quad (48)$$

hence each round sphere $S_{T,R}$ in Minkowski spacetime M_0 is a rigid sphere. Using Poincaré symmetry group of M_0 , it is easy to check that there is an 8-parameter family of such spheres. Indeed, fixing the value of R , a 7-parameter family remains left. All of them may be obtained from a single sphere, say $S_{0,R}$, by the action of the 10-parameter Poincaré group. Because the three-parameter subgroup of rotations corresponds to internal symmetries of $S_{0,R}$, we are left with 7 parameters only. The parameter R corresponds to the dilation group. Hence, we have $10 - 3 + 1 = 8$.

In Euclidean three-space (represented by a slice $\{x^0 = 0\}$ in M_0) the family of rigid spheres reduces to four-parameter family of such spheres, where $4 = 3 + 1$ – three translations plus dilation (or similarity transformations minus rotations $4 = 7 - 3$). Each round sphere in Euclidean three-space is a rigid sphere because its mean extrinsic curvature $k = -\frac{2}{R} \in \mathcal{M}^4$.

3 Existence of rigid spheres in a Riemannian space

Let Σ be a three-dimensional Riemannian manifold. Let $S \subset \Sigma$ be a two-manifold diffeomorphic to the unit sphere $S^2 \subset \mathbb{R}^3$. We consider the following problems: 1) Can we deform S in such a way that the resulting submanifold becomes a rigid sphere? 2) How many of such deformations exist in a vicinity of S ?

To parameterize these deformations we introduce in a neighbourhood of S a Gaussian system of coordinates (u, x^A) . Here, by (x^A) , $A = 1, 2$, we denote any coordinate system on S , whereas u is the arc-length parameter along the “ $\{x^A = \text{const.}\}$ ” geodesics starting orthogonally from S . The three-metric takes, therefore, the form

$$g = du^2 + g_{AB}(u, x^A) dx^A dx^B . \quad (49)$$

Suppose, moreover, that coordinates $(x^A) = (\vartheta, \varphi)$ are conformal and equilibrated on S . This means that we have

$$\dot{g}_{AB} dx^A dx^B = \psi \cdot (\sigma_{AB} dx^A dx^B) = \psi \cdot (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) , \quad (50)$$

where

$$\dot{g}_{AB} := g_{AB}(0, x^A) \quad (51)$$

is the induced two-metric on S , σ is the “round” two-metric on the Euclidean unit sphere:

$$\sigma_{AB} dx^A dx^B = d\vartheta^2 + \sin^2 \vartheta d\varphi^2 , \quad (52)$$

and the function ψ is dipole-free ($\psi^{\text{d}} = 0$). Second fundamental form of S is given by:

$$\dot{k}_{AB} = -\frac{1}{2} g_{AB,u} . \quad (53)$$

Its trace does not need to belong to the space \mathcal{M}^4 of mono-dipole-like functions, i.e. the surface S does not need to be a rigid sphere. We are looking for such deformations of S , for which the resulting surface fulfills already the rigidity condition.

Any deformation of S which is sufficiently small may be uniquely parameterized by a function $\tau = \tau(x^A)$, such that the deformed surface S_τ is given by:

$$S_\tau = \{(u, x^A) \mid u = \tau(x^A)\} . \quad (54)$$

The surface S_τ carries the induced metric:

$$g|_{S_\tau} = [d\tau(x^A)]^2 + g_{AB}(\tau(x^C), x^C) dx^A dx^B =: g_{AB}(x^C) dx^A dx^B , \quad (55)$$

where

$$g_{AB}(x^C) = (\partial_A \tau)(\partial_B \tau) + g_{AB}(\tau(x^C), x^C) . \quad (56)$$

Here, we use the same coordinate system (x^A) , which was previously used for S . However, these coordinates do not need to be neither conformally spherical nor

equilibrated. To verify that the deformation τ was successful, i.e. that S_τ is a rigid sphere, we have to pass to an equilibrated system of spherical coordinates, say \hat{x}^A , on S_τ . To make this choice unique, we use the north pole: $\mathbf{n} := \{\vartheta = 0\}$, and the ‘‘Gulf of Guinea’’: $\mathbf{m} := \{\vartheta = \frac{\pi}{2}; \varphi = 0\}$ to get rid of the rotation non-uniqueness (cf. Theorem 2). This way we obtain an equilibrated version \hat{g}_{AB} of the metric (56). Finally, we calculate the extrinsic curvature k and check whether or not its wave part $k^{\mathbf{w}}(S_\tau)$ satisfies condition $k^{\mathbf{w}}(S_\tau) = 0$.

The idea of our paper may, therefore, be sketched as follows. We begin with a metric (49) which is of the class $C^{(k,\alpha)}$. The above construction defines a continuous mapping:

$$C_{md}^{(k+1,\alpha)} \times C_w^{(k+1,\alpha)} \ni (\tau^{\mathbf{md}}, \tau^{\mathbf{w}}) = \tau \longrightarrow F(\tau) := k^{\mathbf{w}} \in C_w^{(k-1,\alpha)}. \quad (57)$$

Indeed, the resulting metric in a neighbourhood of S_τ is obtained from g and the first derivatives of τ . The function τ being of the class $C^{(k+1,\alpha)}$, the metric obtained this way is again of the class $C^{(k,\alpha)}$. Due to Theorem 2, its equilibrated version \hat{g}_{AB} is again of the same class. Finally, the extrinsic curvature k is obtained using first derivatives of this metric. Hence, the result is of the class $C^{(k-1,\alpha)}$ and the entire procedure is continuous.

Now, rigid spheres are those, which satisfy equation:

$$F(\tau) = 0. \quad (58)$$

We are going to prove that, for a generic metric g , the above equation defines an implicit function:

$$\mathcal{M}^4 \equiv C_{md}^{(k+1,\alpha)} \ni \tau^{\mathbf{md}} \longrightarrow H(\tau^{\mathbf{md}}) \in C_w^{(k+1,\alpha)}, \quad (59)$$

such that

$$F(\tau^{\mathbf{md}}, H(\tau^{\mathbf{md}})) \equiv 0, \quad (60)$$

or, equivalently, that $S_{(\tau^{\mathbf{md}}, H(\tau^{\mathbf{md}}))}$ is a rigid sphere. The main result of our paper follows as a corollary:

Theorem 3. *Generically, there exists a four-parameter family of rigid spheres in a neighbourhood of a given two-sphere $S \subset \Sigma$, corresponding to the four-parameter family of mono-dipole functions $\tau^{\mathbf{md}}$ on S .*

3.1 Infinitesimal deformations of spheres

To prove existence of the implicit function (60) it is sufficient to show that, given a mono-dipole deformation $\tau^{\mathbf{md}}$, the partial derivative of F with respect to the ‘‘wave-like’’ deformation $\tau^{\mathbf{w}}$ is an isomorphism of $C_w^{(k+1,\alpha)}$ onto $C_w^{(k-1,\alpha)}$.

For this purpose, we analyze the infinitesimal, linear version of the construction discussed above. Consider, therefore, a transversal deformation $\tau = \tau(x^A)$ of $S \subset \Sigma$ and a small deformation parameter ε :

$$S \rightsquigarrow S_\tau = \{(u, x^A) \mid u = \varepsilon \tau(x^A)\}. \quad (61)$$

Under such transformation the induced metric changes in the following way:

$$g_{AB} - \dot{g}_{AB} = -2\varepsilon\tau\dot{k}_{AB} + O(\varepsilon^2) . \quad (62)$$

Even if the initial system of coordinates was equilibrated, the transformed metric does not need to be conformally spherical. The non-sphericity of the metric must be, therefore, compensated by a change of coordinates. Its infinitesimal version is described by a tangential (with respect to S) deformation

$$\widehat{x}^A = x^A - \varepsilon\xi^A . \quad (63)$$

Under such coordinate transformation the metric changes as follows:

$$\widehat{g}_{AB} = g_{AB} - \mathcal{L}_{2\varepsilon\vec{\xi}} g_{AB} , \quad (64)$$

where the last term represents the Lie derivative of the metric g_{AB} with respect to the vector field “ $-\varepsilon\xi^A$ ” on S . But, according to (62), the difference between g_{AB} and \dot{g}_{AB} is already of the first order in ε . Hence, if we replace it by the Lie derivative of the metric \dot{g}_{AB} , the error will be of the second order in ε . Using the Killing formula for the Lie derivative of the metric we finally obtain:

$$\widehat{g}_{AB} = g_{AB} + 2\varepsilon\xi_{(A||B)} + O(\varepsilon^2) , \quad (65)$$

and the covariant derivative $_{||A}$ is taken with respect to the original metric \dot{g}_{AB} . Hence, we have:

$$\widehat{g}_{AB} - \dot{g}_{AB} = -2\varepsilon\tau\dot{k}_{AB} + 2\varepsilon\xi_{(A||B)} + O(\varepsilon^2) . \quad (66)$$

Let us decompose the above equation into the trace and the trace-free parts, calculated with respect to \dot{g}_{AB} (we omit the terms of order ε^2 and higher):

$$\widehat{g}_{AB} - \dot{g}_{AB} = \left(\varepsilon\xi^C_{||C} - \varepsilon\tau\dot{k} \right) \dot{g}_{AB} - 2\varepsilon\tau\dot{\kappa}_{AB} + 2\varepsilon \left(\xi_{(A||B)} - \frac{1}{2}\xi^C_{||C}\dot{g}_{AB} \right) , \quad (67)$$

where

$$\dot{\kappa}_{AB} := \dot{k}_{AB} - \frac{1}{2}\dot{k}\dot{g}_{AB} \quad (68)$$

is the traceless part of \dot{k}_{AB} . We want \widehat{g}_{AB} to be conformally spherical, i.e. $\widehat{g}_{AB} = \alpha \cdot \dot{g}_{AB}$. This implies:

$$\left(1 - \varepsilon\tau\dot{k} - \alpha + \varepsilon\xi^C_{||C} \right) \dot{g}_{AB} - 2\varepsilon\tau\dot{\kappa}_{AB} + 2\varepsilon\xi_{(A||B)} - \varepsilon\xi^C_{||C}\dot{g}_{AB} = 0 . \quad (69)$$

The trace part of this equation defines uniquely the value of α :

$$\alpha = 1 - \varepsilon\tau\dot{k} + \varepsilon\xi^C_{||C} , \quad (70)$$

whereas the trace-free part reduces to:

$$\xi_{A||B} + \xi_{B||A} - \xi^C_{||C} \dot{g}_{AB} = 2\tau \dot{\kappa}_{AB} . \quad (71)$$

It is convenient to rewrite equation (71) in terms of the “round” unit-sphere geometry σ_{AB} . For this purpose we use the following conventions: components of a vector (i.e. an object having *upper indices*) are the same in both geometries σ_{AB} and $\dot{g}_{AB} = \psi \sigma_{AB}$. Components of a co-vector (*lowered indices*) are denoted as follows:

$$\xi_A^\sigma = \sigma_{AB} \xi^B , \quad \xi_A = \dot{g}_{AB} \xi^B = \psi \sigma_{AB} = \psi \xi_A^\sigma . \quad (72)$$

The covariant derivative with respect to σ_{AB} will be denoted by $_{||A}$, e.g. $\xi_{A||B}^\sigma$. Equation (71) can be easily rewritten as:

$$\xi_{A||B}^\sigma + \xi_{B||A}^\sigma - \xi^C_{||C} \sigma_{AB} = 2 \frac{\tau}{\psi} \dot{\kappa}_{AB} . \quad (73)$$

The left-hand side of this equation defines a mapping from the space of vector fields on the unit sphere to the space of trace-free rank 2 tensor fields. The kernel of this mapping consists of the dipole fields². The “Fredholm alternative” argument shows that the operator on the left-hand side defines an isomorphism between the space of dipole-free vector fields on the unit sphere and the space of trace-free rank 2 tensor fields (see also [7]). This isomorphism (in metric σ) will be denoted by i_{12} . Hence, the wave part of ξ^A is implied uniquely by equation (73) (see Appendix):

$$\xi^{\mathbf{w}A} = i_{12}^{-1} \left(2 \frac{\tau}{\psi} \dot{\kappa}_{AB} \right) , \quad (74)$$

whereas the dipole part of ξ^A , i.e. the field $\xi^{\mathbf{d}A}$, remains arbitrary.

The above choice of the wave-like component of the tangential deformation $\xi^{\mathbf{w}A}$ guarantees that the new coordinate system \hat{x}^A is conformally spherical. We would like it to be also: 1) equilibrated and 2) satisfying conditions related to the two fixed points \mathbf{n} and \mathbf{m} . These conditions mean that the field ξ has to vanish at the north pole \mathbf{n} and that its φ -component vanishes at \mathbf{m} . The above $3 + 3 = 6$ conditions fix uniquely the total dipole-part of the tangential (to S) deformation ξ^A . This way the continuous mapping which assigns uniquely the tangential deformation ξ^A to its transversal component τ has been defined.

3.2 The infinitesimal change of the extrinsic curvature

Now, we are going to calculate the infinitesimal change of the wave part $k^{\mathbf{w}}$ of the mean curvature³, i.e. derivative of the mapping (57) with respect to the “wave-

²A vector field on the sphere may be uniquely decomposed into the sum of a gradient and a co-gradient. These two components are represented by the corresponding two scalar functions: the divergence and the curl. The multipole expansion of a vector field is uniquely defined by the multipole expansion of these two functions.

³First variations of the *total* mean curvature k is known in the literature as the second variations of area, cf. e.g. [3]. See also discussion in the Appendix.

like” deformation $\tau^{\mathbf{w}}$. We have $k = \tilde{g}^{AB} k_{AB}$, where \tilde{g}^{AB} denotes the inverse of the two-metric g_{AB} (whereas g^{AB} denotes the corresponding components of the inverse three-metric.) The simplest way to calculate this change is to use a coordinate system (ω, x^A) , adapted to the deformed surface:

$$\omega = u - \varepsilon \tau(x^A), \quad \text{i.e. } S_\tau = \{\omega = 0\}, \quad (75)$$

and the formula:

$$k_{AB} = \frac{1}{\sqrt{g^{\omega\omega}}} \Gamma_{AB}^\omega. \quad (76)$$

The three-metric g takes now the following form:

$$g = d\omega^2 + 2\varepsilon \tau_{,A} d\omega dx^A + g_{AB} dx^A dx^B + O(\varepsilon^2). \quad (77)$$

This implies $g^{\omega\omega} = 1 + O(\varepsilon^2)$ and, consequently,

$$k_{AB} = \Gamma_{\omega AB} + g^{\omega C} \Gamma_{CAB} + O(\varepsilon^2) = \frac{1}{2} (g_{\omega A||B} + g_{\omega B||A} - g_{AB,\omega}) + O(\varepsilon^2), \quad (78)$$

where we treat the “shift vector” $g_{\omega A} = \varepsilon \tau_{,A}$ as a covector field on S_τ . The first two terms combine to $\varepsilon \tau_{||AB}$, whereas the last one: $g_{AB,\omega}(S_\tau)$ can be approximated by the quantity $g_{AB,\omega}(S) = -2\mathring{k}_{AB}$ plus the derivative of this object. Finally, we have

$$k_{AB} = \mathring{k}_{AB} + \varepsilon \tau \mathring{k}_{AB,u} + \varepsilon \tau_{||AB} + O(\varepsilon^2). \quad (79)$$

Since the derivative $g_{AB,\omega}$ of the metric g_{AB} is described by $-2\mathring{k}_{AB}$, the derivative of its inverse \tilde{g}^{AB} is described by $+2\mathring{k}^{AB}$. Hence, we have:

$$\tilde{g}^{AB} - \mathring{g}^{AB} = 2\varepsilon \tau \mathring{k}^{AB} + O(\varepsilon^2), \quad (80)$$

and, consequently:

$$k = \tilde{g}^{AB} k_{AB} = \mathring{k} + \varepsilon \tau \partial_u \mathring{k} + \varepsilon \tau^{||A}{}_{,A} + O(\varepsilon^2). \quad (81)$$

The quantity $\tau \partial_u \mathring{k} + \tau^{||A}{}_{,A}$ describes already the second variation of area (see Appendix), i.e. the derivative $\nabla_\tau k$. However, to calculate the derivative of the mapping (57), we have to select its wave part $k^{\mathbf{w}}$. For this purpose we have to pass to the conformally spherical, equilibrated coordinates \hat{x}^A , given by formula (63). Infinitesimal change of the scalar function k with respect to this deformation is given by formula:

$$\hat{k} = k - \varepsilon \xi^A k_{,A} + O(\varepsilon^2).$$

Hence, we get:

$$\frac{1}{\varepsilon} (\hat{k} - \mathring{k}) = \tau \partial_u \mathring{k} + \tau^{||A}{}_{,A} - \xi^A \mathring{k}_{,A} + O(\varepsilon), \quad (82)$$

or, equivalently (cf. Appendix),

$$\frac{1}{\varepsilon} (\hat{k} - \mathring{k}) = \tau (R^u{}_u + \mathring{k}^{AB} \mathring{k}_{AB}) + \tau^{||A}{}_{,A} - \xi^A \mathring{k}_{,A} + O(\varepsilon). \quad (83)$$

3.3 Proof of the Theorem 3

The last formula gives, finally, the value of the derivative of the mapping (57). When restricted to the subspace of wave (i.e. mono-dipole-free) deformations, it gives us:

$$C_w^{(k+1,\alpha)} \ni \tau \mapsto \left[\tau \left(R^u_u + \overset{\circ}{k}{}^{AB} \overset{\circ}{k}{}_{AB} \right) + \tau^{\parallel A}{}_A - \xi^A \overset{\circ}{k}{}_{,A} \right]^{\mathbf{w}} \in C_w^{(k-1,\alpha)} . \quad (84)$$

The above linear operator is, obviously, continuous. In particular, the vector field ξ^A is given by formula (74), together with the accompanying vanishing conditions at \mathbf{n} and \mathbf{m} .

If the space Σ is flat (Euclidean) and S is a standard (rigid) sphere of radius r , then we have:

$$\overset{\circ}{g}_{AB} = r^2 \sigma_{AB}; \quad \overset{\circ}{k}_{AB} = -r \sigma_{AB}; \quad \overset{\circ}{k}_{,A} = 0; \quad R^u_u = 0. \quad (85)$$

Hence, the above operator reduces to:

$$\begin{aligned} \tau^{\mathbf{w}} &\mapsto \left[\tau \left(R^u_u + \overset{\circ}{k}{}^{AB} \overset{\circ}{k}{}_{AB} \right) + \tau^{\parallel A}{}_A - \xi^A \overset{\circ}{k}{}_{,A} \right]^{\mathbf{w}} = \frac{1}{r^2} [(\Delta_\sigma + 2)\tau]^{\mathbf{w}} \\ &= \frac{1}{r^2} (\Delta_\sigma + 2)(\tau^{\mathbf{w}}) , \end{aligned} \quad (86)$$

which is obviously an invertible mapping from $C_w^{(k+1,\alpha)}$ to $C_w^{(k-1,\alpha)}$. But the mapping (86) depends in a continuous way upon the geometry (metric and curvature) of S . This implies that it remains invertible for sufficiently small deformations of the geometry.

Suppose now, that Σ is asymptotically flat, i.e. there is a coordinate chart (x^k) which covers the exterior of a compact domain $D \subset \Sigma$, such that

$$g_{kl} = \delta_{kl} + h_{kl} ,$$

where h vanishes sufficiently fast at infinity. This means that the above statement remains true for “coordinate spheres” defined as follows:

$$S_{\vec{x}_0, R} := \left\{ x \in \mathbb{R}^3 \mid \sum_{i=1}^3 (x^i - x_0^i)^2 = R^2 \right\} ,$$

if the radius R is sufficiently big. We conclude that $\Sigma \setminus D$ admits a four-parameter family of rigid spheres, similarly as in the case of the flat metric.

4 Conclusions

The main technical ingredient of this paper is the intrinsic, coordinate invariant definition of the “multipole expansion” of a function defined on a Riemannian two-manifold, diffeomorphic with S^2 . This enables us to select a finite-dimensional

family of “rigid spheres”. The dipole part $k^{\mathbf{d}}$ of the curvature parameterizes the position of the center of such a sphere with respect to the center of mass. In particular, $k^{\mathbf{d}} = 0$ corresponds to the spheres, which are centered at the center of mass. Properties of such a foliation have been analyzed in [4]. General topologically spherical coordinates, having property that surfaces $\{r = \text{const.}\}$ are rigid, do not admit *supertranslations* ambiguity at space infinity. This way symmetries of the “tangent space at infinity” reduce to a finite-dimensional one. The 4D version of our results, valid for a generic four-dimensional Lorenzian spacetime, which will be presented in the subsequent paper, will do the same job for the symmetry group of the Scri.

Acknowledgements

This research was supported by Polish Ministry of Science and Higher Education grant Nr N N201 372736. SŁ was supported by Foundation for Polish Science.

A Appendix

A.1 The dipole part of traceless symmetric part

The kernel of the mapping

$$\xi_A^\sigma \mapsto \xi_{A\lambda B}^\sigma + \xi_{B\lambda A}^\sigma - \sigma^{CD} \xi_{C\lambda D}^\sigma \sigma_{AB}$$

defined by the left-hand side of the formula (73) consists of the dipole fields. This is a simple consequence of the following observations.

- In case of the unit sphere the Hodge decomposition $\xi = d\alpha + \delta\beta + h$ of the covector ξ on a compact manifold does not contain the harmonic part, i.e. harmonic one-form h vanishes ($dh = 0 = \delta h$ implies $h = 0$). The topology of the unit sphere (triviality of the corresponding cohomology class) cancels the harmonic part and we can always represent ξ as follows

$$\xi_A = \alpha_{,A} + \varepsilon_A^B \beta_{,B}, \quad (87)$$

where functions α and β are defined up to a constant but their gradients are unique.

- The purely dipole covector ξ simply means that the potentials α and β are purely dipole functions: $\alpha = a_i X^i$, $\beta = b_i X^i$, where a_i , b_i are real constants.
- Direct computation for dipole functions X^i enables one to check the following identity: $X_{\lambda AB}^i = -X^i \sigma_{AB}$, hence for any dipole function α

$$\alpha_{\lambda AB} = -\alpha \sigma_{AB}. \quad (88)$$

- Formulae (87) and (88) give

$$\xi_{A\mathbb{L}B} = -\alpha\sigma_{AB} - \beta\varepsilon_{AB},$$

hence the traceless symmetric part of $\xi_{A\mathbb{L}B}$ vanishes.

A.2 The isomorphism between covector fields and symmetric traceless tensors on (S^2, σ_{AB})

Let us consider the following diagram:

$$\begin{array}{ccccccccc} V_{k+2}^0 \oplus V_{k+2}^0 & \xrightarrow{i_{01}} & V_{k+1}^1 & \xrightarrow{i_{12}} & V_k^2 & \xrightarrow{i_{21}} & V_{k-1}^1 & \xrightarrow{i_{10}} & V_{k-2}^0 \oplus V_{k-2}^0 \\ \downarrow Fl & & \downarrow \hat{\cdot} & & \downarrow \hat{\cdot} & & \downarrow \hat{\cdot} & & \downarrow Fl \\ V_{k+2}^0 \oplus V_{k+2}^0 & \xrightarrow{i_{01}} & V_{k+1}^1 & \xrightarrow{i_{12}} & V_k^2 & \xrightarrow{i_{21}} & V_{k-1}^1 & \xrightarrow{i_{10}} & V_{k-2}^0 \oplus V_{k-2}^0 \end{array}$$

where the mappings and the spaces are defined as follows:

$$i_{01}(f, g) = f_{\mathbb{L}A} + \varepsilon_A^B g_{\mathbb{L}B},$$

$$i_{12}(v) = v_{A\mathbb{L}B} + v_{B\mathbb{L}A} - \sigma_{AB} v^C{}_{\mathbb{L}C},$$

$$i_{21}(\chi) = \chi_A^B{}_{\mathbb{L}B},$$

$$i_{10}(v) = (v_{\mathbb{L}A}^A, \varepsilon^{AB} v_{A\mathbb{L}B}),$$

$$Fl(f, g) = (g, f), \quad \hat{v}_A = \varepsilon_A^B v_B, \quad \hat{\chi}_{AB} = \varepsilon_A^C \chi_{CB},$$

V_k^0 – scalars on S^2 belonging to Hölder space $C^{k,\alpha}$,

V_k^1 – covectors on S^2 belonging to Hölder space $C^{k,\alpha}$,

V_k^2 – symmetric traceless tensors on S^2 belonging to Hölder space $C^{k,\alpha}$.

Denote by Δ_σ the Laplace operator on S^2 and by SH^l the space of spherical harmonics of degree l , ($f \in SH^l \iff \Delta_\sigma f = -l(l+1)f$). The following equality

$$i_{10} \circ i_{21} \circ i_{12} \circ i_{01} = \Delta_\sigma(\Delta_\sigma + 2)$$

shows that if we restrict ourselves to the spaces $\bar{V}^0 := V^0 \ominus [SH^0 \oplus SH^1] = (I - P_{md})V^0$ ($\Delta_\sigma(\Delta_\sigma + 2)\bar{V}^0 = \bar{V}^0$) and $\bar{V}^1 = V^1 \ominus [i_{01}(SH^1)]$ ($(\Delta_\sigma + I)\bar{V}^1 = \bar{V}^1$) then all the mappings in the above diagram become isomorphisms.

A.2.1 Integral operators, generalized Green's functions

Solution of the Helmholtz equation on a unit sphere S^2 :

$$[\Delta_\sigma + l(l+1)] \Psi_l(n) = \Phi(n), \quad n \in S^2$$

is given (see e.g. [11]) in terms of the generalized Green's function \bar{G}_l as follows:

$$\Psi_l(n) = \int_{S^2} \bar{G}_l(n, n') \Phi(n') d^2 n'. \quad (89)$$

Here $n = \mathbf{D}(\vartheta, \varphi)$ given by (18) and $d^2 n = d\sigma$. The solution $\Psi_l(n)$ is automatically orthogonal to the space SH^l (the kernel of Helmholtz operator $\Delta_\sigma + l(l+1)$) because Green's function is orthogonal to this space. In our case we need to write the inverse of the operator $\Delta_\sigma(\Delta_\sigma + 2)$ as a double integral with the corresponding kernels \bar{G}_l for $l = 0$ and $l = 1$. More precisely, the solution g of the equation $\Delta_\sigma(\Delta_\sigma + 2)g = f$ (with $P_{md}f = 0$) is given in the following form:

$$\begin{aligned} g(n) &= P_w \int_{S^2} \bar{G}_0(n, n'') \cdot \\ &\quad \cdot \left[\int_{S^2} \bar{G}_1(n'', n') f(n') d^2 n' - \frac{1}{4\pi} \int_{S^2 \times S^2} \bar{G}_1(m, n') f(n') d^2 n' d^2 m \right] d^2 n'' \\ &= \int_{S^2} \bar{G}_0(n, n'') \left[\int_{S^2} \bar{G}_1(n'', n') f(n') d^2 n' \right] d^2 n'', \end{aligned} \quad (90)$$

where the projection operator P_w provides orthogonality⁴ of g to the space $SH^0 \oplus SH^1$. The generalized Green's function written in a standard form:

$$\bar{G}_l(n, n') = \sum_{i=0, i \neq l}^{\infty} \sum_{m=-i}^i \frac{Y_{im}(n) \overline{Y_{im}(n')}}{l(l+1) - i(i+1)}, \quad Y_{im} \in SH^i,$$

can be simplified as follows (cf. [11]):

$$\begin{aligned} \bar{G}_l(n, n') &= \frac{1}{4\pi} P_l(n \cdot n') \left[\ln \frac{1 - n \cdot n'}{2} + c_l \right] + \frac{1}{2\pi} \sum_{i=0}^{l-1} \frac{2i+1}{(l-i)(l+i+1)} P_i(n \cdot n'), \\ c_l &:= \frac{1}{2l+1} - 2 \sum_{i=0}^{l-1} (-1)^{l+i} \frac{2i+1}{(l-i)(l+i+1)}, \end{aligned}$$

Y_{im} – spherical harmonics (orthonormal basis in SH^i), $n \cdot n' \in [-1, 1]$ is a scalar product of unit vectors in \mathbb{R}^3 and $P_l(x) := \frac{1}{2^l l!} (x^2 - 1)^{(l)}$ is the Legendre polynomial.

A.3 Second variation of area

The Gaussian coordinates (49) and the definition of the Riemann tensor gives

$$R^u{}_{AuB} = \mathring{k}_{AB,u} + \mathring{k}_A{}^C \mathring{k}_{BC}. \quad (91)$$

⁴The above integral operators do not mix the wave part with the mono-dipole part of a function. This means that $P_w f = f$ implies $P_w(G_1 * f) = G_1 * f$ and $P_w(G_0 * f) = G_0 * f$.

This leads to

$$k_{AB} = \mathring{k}_{AB} + \varepsilon \tau (R^u_{AuB} - \mathring{k}_A^C \mathring{k}_{BC}) + \varepsilon \tau_{||AB} + O(\varepsilon^2). \quad (92)$$

Taking the trace (and using (80)), we obtain:

$$k = \mathring{k} + \varepsilon \tau (R^u_u + \mathring{k}^{AB} \mathring{k}_{AB}) + \varepsilon \tau^{||A}_A + O(\varepsilon^2). \quad (93)$$

The formulae (81) and (93) are equivalent because of the Gauss-Codazzi equations:

$$2\partial_u \mathring{k} = R(g_{kl}) - R(\mathring{g}_{AB}) + \mathring{k}^{AB} \mathring{k}_{AB} + \mathring{k}^2, \quad (94)$$

$$2R^u_u = R(g_{kl}) - R(\mathring{g}_{AB}) - \mathring{k}^{AB} \mathring{k}_{AB} + \mathring{k}^2, \quad (95)$$

where $R(g_{kl})$ and $R(\mathring{g}_{AB})$ are scalar of curvatures of the three-metric g_{kl} and the two-metric \mathring{g}_{AB} , respectively. Obviously (81), (93) are the first variations of the mean curvature k , which in the literature (see e.g. [3]) are known as the second variations of area. They are usually presented in the following equivalent form:

$$k - \mathring{k} = \frac{\varepsilon}{2} \left[\left(R(g_{kl}) - R(\mathring{g}_{AB}) + \mathring{k}^{AB} \mathring{k}_{AB} + \mathring{k}^2 \right) \tau + 2\tau^{||A}_A \right] + O(\varepsilon^2). \quad (96)$$

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